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I. Definition of the Main Problem

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ANALYTICAL LUNAR EPHEMERIS

I. DEFINITION OF THE MAIN PROBLEM

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SUMMARY

A new set of phase variables is proposed and justified to develop automatically by computer the solar terms in Lunar Theory.

The dependence of the perturbation function on the mass ratio Moon/(Earth + Moon) is completely elucidated. A recursive procedure is proposed to develop that function so as to keep explicit all its d'Alembert characteristics.

The perturbation series obtained by computer is compared with Delaunay's development.

By an Analytical Lunar Ephemeris, we mean a general approximate solution applying to any single moon whose orbit satisfies certain *a priori* assumptions; the coordinates and velocity components, or any other state variable, come out as literal expressions in terms of initial constants and of dynamical parameters carried throughout the theory as symbols to which nowhere is given a numerical value. An Analytical Lunar Ephemeris presents definite advantages. The degree of accuracy to which the theory is constructed is given an explicit definition in reference to the physical assumptions and the size of the parameters involved. It is immediately applicable to any single moon in our solar system. It is arranged in such a way that any small change caused by improved data on the values of the constants can be made easily without requiring to go over the whole of the work again. In the operations of fitting the theory to the observations, it readily provides the true partial differential coefficients relative to the model on which it was constructed. It is the only sure way to decide whether the residuals O-C lie in inadequacies of the nominal model or in inaccuracies of the solution. Finally a numerical or operational mistake in the course of building an algebraic expression is very easily traceable.

On the other hand, the number of terms which have been found necessary to secure a degree of accuracy commensurate with that of observations is very large, and, without the help of a computer, it becomes an impossible task to obtain them with any degree of certainty and thoroughness.

The formation of an Analytical Lunar Ephemeris may be roughly divided into three stages. At first, the three bodies Sun, Earth and Moon are taken as mass particles; the barycenter of the pair Earth-Moon is supposed to move round the Sun in a given Keplerian orbit; all displacements of this orbit and of the Moon's orbit caused by any other source than the Sun and the Earth as mass particles are ignored. This initial stage is called the Main Problem, and its solution constitutes the Solar Part of an Analytical Lunar Ephemeris. In the second phase, secondary perturbations are investigated. The algebraic construction may proceed, at least in principle, without any knowledge of initial conditions beyond a crude estimate of the relative magnitude among some of the constants involved. The third stage would then be devoted to mounting analytically the partial derivatives with respect to the astronomical constants to be determined from observations, in evaluating the constants numerically and in substituting these numbers in the analytical expressions so as to obtain a Numerical Lunar Ephemeris.

Of the methods which have been proposed to solve analytically the Main Problem, Delaunay's *Théorie du Mouvement de la Lune* must take the first place. His algorithms may not be the best adapted to automated algebraic calculation (Barton 1966), but he has actually carried his construction to a high degree and with a detail greater than any other. Delaunay's formulae offer numerous opportunities for testing subroutines and assessing what new analytical solutions of the Main Problem may achieve.

During the last four years, we have been considering implementing by computer the suggestion of Dr. Brouwer (1961). The algorithm of Von Zeipel which he proposed may appear troublesome, chiefly from its inability to produce the eliminating canonical transformations and their inverses in an explicit form. Mersman (1969) has significantly developed the capabilities of Von Zeipel's method. Yet Mersman's algorithms for unmixing the variables are not recursive, which makes it difficult to involve them in general codes where the order of a theory enters as dummy argument. Some time ago (Deprit 1969), it has been shown how the difficulties raised by Von Zeipel's method might be avoided by the use of Lie transforms. At the same time, our computer techniques (Danby et al. 1965, Rom 1970) have undergone a complete overhauling to accommodate divisors made of products of linear combinations of mean motions.

It is intended to combine these two techniques--the canonical perturbation theory based on Lie transforms and the processing of Echeloned Series--so as to completely solve the Main Problem of an Analytical Lunar Emphemeris.

1. THE PERTURBATION FUNCTION

Let $\underset{\sim}{u}_0$ (*resp.* $\underset{\sim}{u}_1, \underset{\sim}{u}_2$) designate the position of the Earth (*resp.* of the Moon, of the Sun); to these position vectors are associated the impulsions

$$\underset{\sim}{U}_i = m_i \dot{\underset{\sim}{u}}_i, \qquad (i=0,1,2) \tag{1}$$

in which m_0 (*resp.* m_1, m_2) indicates the mass of the Earth (*resp.* of the Moon, of the Sun). The mutual distances being the functions

$$r_{0,1} = ||\underset{\sim}{u}_0 - \underset{\sim}{u}_1||, \qquad r_{0,2} = ||\underset{\sim}{u}_0 - \underset{\sim}{u}_2||, \qquad r_{1,2} = ||\underset{\sim}{u}_1 - \underset{\sim}{u}_2||, \tag{2}$$

this problem of three bodies is described by the Hamiltonian

$$\mathcal{H} = \frac{1}{2} \sum_{0 \leq i \leq 2} \frac{1}{m_i} ||\underset{\sim}{U}_i||^2 - k^2 \sum_{(i,j)} \frac{m_i m_j}{r_{ij}} . \tag{3}$$

Positions $(\underset{\sim}{u}_0, \underset{\sim}{u}_1, \underset{\sim}{u}_2)$ in the inertial frame are transferred to positions $(\underset{\sim}{x}_0, \underset{\sim}{x}_1, \underset{\sim}{x}_2)$ in a chain of barycentric coordinate systems according to the formulas

$$\begin{aligned}
\underset{\sim}{x}_1 &= \underset{\sim}{u}_1 - \underset{\sim}{u}_0, \\
\underset{\sim}{x}_2 &= \underset{\sim}{u}_2 - (1-\sigma_1)\underset{\sim}{u}_0 - \sigma_1 \underset{\sim}{u}_1, \\
\underset{\sim}{x}_0 &= \sigma_2 \underset{\sim}{u}_2 + (1-\sigma_2)[(1-\sigma_1)\underset{\sim}{u}_0 + \sigma_1 \underset{\sim}{u}_1]
\end{aligned} \tag{4}$$

wherein

$$\sigma_1 = \frac{m_1}{m_0+m_1}, \qquad \sigma_2 = \frac{m_2}{m_0+m_1+m_2} . \tag{5}$$

The inverse mapping

$$\begin{aligned} \underset{\sim}{u}_0 &= \underset{\sim}{x}_0 - \sigma_1 \underset{\sim}{x}_1 - \sigma_2 \underset{\sim}{x}_2, \\ \underset{\sim}{u}_1 &= \underset{\sim}{x}_0 + (1-\sigma_1)\underset{\sim}{x}_1 - \sigma_2 \underset{\sim}{x}_2, \\ \underset{\sim}{u}_2 &= \underset{\sim}{x}_0 + (1-\sigma_2)\underset{\sim}{x}_2 \end{aligned} \tag{6}$$

extends through the implicit definitions

$$\begin{aligned} \underset{\sim}{U}_0 &= (1-\sigma_1)(1-\sigma_2)\underset{\sim}{X}_0 - \underset{\sim}{X}_1^* - \sigma_1 \underset{\sim}{X}_2^*, \\ \underset{\sim}{U}_1 &= \sigma_1(1-\sigma_2)\underset{\sim}{X}_0 + \underset{\sim}{X}_1^* - (1-\sigma_1)\underset{\sim}{X}_2^*, \\ \underset{\sim}{U}_2 &= \sigma_2 \underset{\sim}{X}_0 + \underset{\sim}{X}_2^* \end{aligned} \tag{7}$$

into a homogeneous linear canonical transformation. The vector $\underset{\sim}{x}_1$ is the geocentric position of the Moon, the vector $\underset{\sim}{x}_2$, the position of the Sun with respect to the barycenter of the couple (Earth-Moon) and the vector $\underset{\sim}{x}_0$, the inertial position of the barycenter of the system (Earth, Moon, Sun). Under this homogeneous canonical extension the Hamiltonian decomposes into the sum

$$\mathcal{H} = \mathcal{C}_0 + \frac{m_0 m_1}{m_0+m_1}\mathcal{H}_1 + \frac{(m_0+m_1)m_2}{m_0+m_1+m_2}\mathcal{H}_2 + \mathcal{P}^* \tag{8}$$

whose terms are

$$\mathcal{C}_0 = \frac{1}{2}(m_0+m_1+m_2)\,||\,\underset{\sim}{X}_0\,||^2, \tag{9_1}$$

$$\mathcal{H}_1 = \frac{1}{2}\,||\,\frac{m_0+m_1}{m_0 m_1}\,\underset{\sim}{X}_1^*\,||^2 - \frac{k^2(m_0+m_1)}{r_1}, \tag{9_2}$$

$$\mathcal{H}_2 = \frac{1}{2}\,||\,\frac{m_0+m_1+m_2}{(m_0+m_1)m_2}\,\underset{\sim}{X}_2^*\,||^2 - \frac{k^2(m_0+m_1+m_2)}{r_2}, \tag{9_3}$$

$$\mathcal{P}^* = k^2 m_2[m_0(\frac{1}{r_2} - \frac{1}{r_{0,2}}) + m_1(\frac{1}{r_2} - \frac{1}{r_{0,1}})]. \tag{9_4}$$

The distance functions entering the Hamiltonian are now

$$r_1 = r_{0,1} = ||\underset{\sim}{x}_1||, \tag{10_1}$$

$$r_2 = ||\underset{\sim}{x}_2||, \tag{10_2}$$

$$r_{0,2} = r_2 |1 + 2s\sigma_1 \frac{r_1}{r_2} + \sigma_1^2 (\frac{r_1}{r_2})^2|^{1/2}, \tag{10_3}$$

$$r_{1,2} = r_2 |1 - 2s(1-\sigma_1)\frac{r_1}{r_2} + (1-\sigma_1)^2 (\frac{r_1}{r_2})^2|^{1/2}, \tag{10_4}$$

provided the elongation function s is defined by the relation

$$r_1 r_2 s = \underset{\sim}{x}_1 \cdot \underset{\sim}{x}_2. \tag{11}$$

Following Poincaré (1909, p. 5), we define the impulsions per unit of mass

$$\underset{\sim}{X}_1 = \frac{m_0+m_1}{m_0 m_1} \underset{\sim}{X}_1^*, \tag{12_1}$$

$$\underset{\sim}{X}_2 = \frac{m_0+m_1+m_2}{(m_0+m_1)m_2} \underset{\sim}{X}_2^*. \tag{12_2}$$

By neglecting the contribution made by the terms of $\mathcal{P}^*$ that depend on the Sun's position $\underset{\sim}{x}_2$, we account for the motion of the Sun through the Hamiltonian

$$\mathcal{H}_2 = \frac{1}{2} ||\underset{\sim}{X}_2||^2 - \frac{k^2(m_0+m_1+m_2)}{r_2}. \tag{13}$$

In other words, we assume that the Sun moves on a fixed ellipse having a focus at the barycenter of the couple (Earth, Moon); let a_2 and e_2 designate respectively the semi-major axis and the eccentricity of the solar orbit. The Sun's mean motion satisfies Kepler's third law, i.e.

$$n_2^2 a_2^3 = k^2(m_0+m_1+m_2). \tag{14}$$

From now on the orbital plane of the Sun is taken as the reference plane (x,y). If g_2 is the argument of perigee of the Sun reckoned from the x-axis and f_2 its true anomaly, then the coordinates of the Sun's position $\underset{\sim}{x}_2$ are

$$x_2 = r_2 \cos(f_2 + g_2), \tag{15_1}$$

$$y_2 = r_2 \sin(f_2 + g_2), \tag{15_2}$$

$$z_2 = 0. \tag{15_3}$$

Under this approximation, the motion of the Moon is described by the Hamiltonian

$$\mathcal{M} \equiv \mathcal{M}(\underset{\sim}{x}_1, \underset{\sim}{X}_1; \underset{\sim}{x}_2(t)) = \mathcal{H}_1(\underset{\sim}{x}_1, \underset{\sim}{X}_1) + \mathcal{P}(\underset{\sim}{x}_1, \underset{\sim}{x}_2(t)) \tag{16_1}$$

with

$$\mathcal{H}_1 = \frac{1}{2} \|\underset{\sim}{X}_1\|^2 - \frac{k^2(m_0+m_1)}{r_1}, \tag{16_2}$$

$$\mathcal{P} = \frac{m_0+m_1}{m_0 m_1} \mathcal{P}^* = k^2 m_2 \left[\frac{1}{\sigma_1}\left(\frac{1}{r_2} - \frac{1}{r_{0,2}}\right) + \frac{1}{1-\sigma_1}\left(\frac{1}{r_2} - \frac{1}{r_{1,2}}\right)\right]. \tag{16_3}$$

But, in application of the binomial law under the assumption that $r_1 < r_2$,

$$\frac{1}{r_{0,2}} = \frac{1}{r_2} \sum_{n \geq 0} (-\sigma_1)^n \left(\frac{r_1}{r_2}\right)^n P_n(s),$$

$$\frac{1}{r_{1,2}} = \frac{1}{r_2} \sum_{n \geq 0} (-\sigma_1)^n \left(\frac{r_1}{r_2}\right)^n P_n(s)$$

where, for any $n \geq 0$, P_n is Legendre's polynomial of degree n. Consequently

$$\mathcal{P} = -\frac{k^2 m_2}{r_2} \sum_{n \geq 2} \left[(1-\sigma_1)^{n-1} - (-\sigma_1)^{n-1}\right] \left(\frac{r_1}{r_2}\right)^n P_n(s). \tag{17}$$

2. BASIC PHASE COORDINATES

We substitute Delaunay's phase variables $(\ell_1, g_1, h_1, L_1, G_1, H_1)$ to the Cartesian coordinates $\mathbf{x} = (x_1, y_1, z_1)$ and conjugate momenta $\mathbf{X}_1 = (X_1, Y_1, Z_1)$. Then we modify them into the set

$$\begin{aligned} \lambda_1 &= \ell_1 + g_1 + h_1, & \Lambda_1 &= L_1, \\ p_1 &= -g_1 - h_1, & P_1 &= L_1 - G_1, \\ q_1 &= -h_1, & Q_1 &= G_1 - H_1. \end{aligned} \tag{18}$$

The phase space of the Main Problem in Lunar Theory will be here coordinatized by this set of modified Delaunay's elements.

In relation with Delaunay's action Λ_1, we define the semi-major axis a_1 and the mean motion n_1 such that

$$\Lambda_1^2 = k^2(m_0 + m_1)a_1, \tag{19$_1$}$$

$$n_1^2 a_1^3 = k^2(m_0 + m_1). \tag{19$_2$}$$

Then we introduce the parameter of lunar parallax

$$\alpha = \frac{a_1}{a_2}, \tag{20}$$

the modified parameter of lunar parallax

$$\alpha_1 = (1-2\sigma_1)\alpha; \tag{21}$$

also we designate by

$$m = n_2/n_1 \tag{22}$$

the ratio of the mean motions of Sun and Moon.

In relation with the norm of the angular momentum G_1, we found convenient to use, beside the eccentricity e_1, a function $E_1 \equiv E_1(L_1,P_1) > 0$ such that

$$2P_1 = L_1E_1^2. \tag{23}$$

From the obvious relation

$$e_1^2 = E_1^2 - \frac{1}{4}E_1^4, \tag{24}$$

there results that

$$\frac{1}{2}E_1^2 = 1 - (1-e_1^2)^{1/2}. \tag{25}$$

In terms of what is sometimes called the anomaly of the eccentricity, namely the angle ϕ_1 such that

$$0 \leq \phi_1 \leq \pi/2, \qquad e_1 = \sin \phi_1,$$

it turns out that

$$E_1 = 2\sin(\phi_1/2).$$

In relation with the component H_1 of the angular momentum normal to the ecliptic, we found convenient to use, beside the inclination I_1, a function $J_1 \equiv J_1(L_1,Q_1) > 0$ such that

$$2Q_1 = L_1J_1^2. \tag{26}$$

It is easily seen that

$$J_1 = 2(1 - e_1^2)^{1/4}\sin\frac{1}{2}I_1 = 2(1 - \frac{1}{2}E_1^2)^{1/2}\sin\frac{1}{2}I_1.$$

Also, we shall often substitute to the angles g_1 and h_1 the linear combinations

$$F_1 = \ell_1 + g_1, \tag{27_1}$$

$$D_1 = \lambda_1 - \lambda_2 = \ell_1 + g_1 + h_1 - \ell_2 - g_2 \tag{27_2}$$

that have been introduced by Delaunay under the names of mean elongation of the Moon to its node and of mean elongation of the Moon to the Sun, respectively.

The development of the Analytical Lunar Ephemeris involves a number of basic differentiations. Reviewing them here will justify partly why we departed from Delaunay's traditional functions e_1 and I_1 in favor of the quantities E_1 and J_1.

From the elementary derivatives

$$\frac{\partial}{\partial L_1} a_1 = \frac{2}{n_1 a_1}, \qquad \frac{\partial}{\partial L_1} n_1 = -\frac{3}{a_1^2},$$

we immediately deduce that

$$\frac{\partial}{\partial L_1} \frac{k^2(m_0+m_1)}{a_1} = -2n_1, \tag{28_1}$$

$$\frac{\partial}{\partial L_1} \frac{1}{n_1} = -3\left[\frac{k^2(m_0+m_1)}{a_1}\right]^{-1}, \tag{28_2}$$

and, for any $j \geq 0$,

$$\frac{k^2(m_0+m_1)}{a_1} \frac{\partial}{\partial L_1} \alpha_1^j = 2jn_1\alpha_1^j, \tag{28_3}$$

$$\frac{k^2(m_0+m_1)}{a_1} \frac{\partial}{\partial L_1} m^j = 3jn_1\alpha_1^j; \tag{28_4}$$

similarly, from the elementary derivatives

$$\frac{\partial}{\partial L_1} E_1 = -\frac{1}{2}\frac{E_1}{L_1}, \qquad \frac{\partial}{\partial L_1} J_1 = -\frac{1}{2}\frac{J_1}{L_1},$$

we calculate that, for any $j \geq 0$,

$$\frac{k^2(m_0+m_1)}{a_1}\frac{\partial}{\partial L_1} E_1^j = -\frac{1}{2} j n_1 E_1^j, \tag{29_1}$$

$$\frac{k^2(m_0+m_1)}{a_1}\frac{\partial}{\partial L_1} J_1^j = -\frac{1}{2} j n_1 J_1^j. \tag{29_2}$$

Also, from

$$\frac{\partial}{\partial P_1} E_1 = \frac{1}{L_1 E_1}, \qquad \frac{\partial}{\partial Q_1} J_1 = \frac{1}{L_1 J_1},$$

we conclude that, for any $j \geq 0$,

$$\frac{k^2(m_0+m_1)}{a_1}\frac{\partial}{\partial P_1} E_1^j = j n_1 E_1^{j-2}, \tag{30_1}$$

$$\frac{k^2(m_0+m_1)}{a_1}\frac{\partial}{\partial Q_1} J_1^j = j n_1 J_1^{j-2}. \tag{30_2}$$

We conclude this sequence of elementary differentiations by a set of simple formulas for the partial derivatives with respect to the actions Λ_1, P_1, Q_1 of the two types of monomials that we shall meet time and again in the development of an Analytical Lunar Ephemeris.

$$\frac{\partial}{\partial L_1}\left[\frac{k^2(m_0+m_1)}{a_1}\,\alpha_1^{j_1} m^{j_2} E_1^{j_3} J_1^{j_4}\right] = (2j_1 + 3j_2 - \tfrac{1}{2}\,j_3 - \tfrac{1}{2}\,j_4 - 2)\alpha_1^{j_1} m^{j_2} E_1^{j_3} J_1^{j_4} n_1, \tag{31$_1$}$$

$$\frac{\partial}{\partial P_1}\left[\frac{k^2(m_0+m_1)}{a_1}\,\alpha_1^{j_1} m^{j_2} E_1^{j_3} J_1^{j_4}\right] = j_3\alpha_1^{j_1} m^{j_2} E_1^{j_3-2} J_1^{j_4} n_1, \tag{31$_2$}$$

$$\frac{\partial}{\partial Q_1}\left[\frac{k^2(m_0+m_1)}{a_1}\,\alpha_1^{j_1} m^{j_2} E_1^{j_3} J_1^{j_4}\right] = j_4\alpha_1^{j_1} m^{j_2} E_1^{j_3} J_1^{j_4-2} n_1, \tag{31$_3$}$$

$$\frac{\partial}{\partial L_1}\left[\frac{k^2(m_0+m_1)}{a_1}\,\alpha_1^{j_1} m^{j_2} E_1^{j_3} J_1^{j_4}\,\frac{1}{n_1}\right] = (2j_1 + 3j_2 - \tfrac{1}{2}\,j_3 - \tfrac{1}{2}\,j_4 + 1)\alpha_1^{j_1} m^{j_2} E_1^{j_3} J_1^{j_4}, \tag{31$_4$}$$

$$\frac{\partial}{\partial P_1}\left[\frac{k^2(m_0+m_1)}{a_1}\,\alpha_1^{j_1} m^{j_2} E_1^{j_3} J_1^{j_4}\,\frac{1}{n_1}\right] = j_3\alpha_1^{j_1} m^{j_2} E_1^{j_3-2} J_1^{j_4}, \tag{31$_5$}$$

$$\frac{\partial}{\partial Q_1}\left[\frac{k^2(m_0+m_1)}{a_1}\,\alpha_1^{j_1} m^{j_2} E_1^{j_3} J_1^{j_4}\,\frac{1}{n_1}\right] = j_4\alpha_1^{j_1} m^{j_2} E_1^{j_3} J_1^{j_4-2}. \tag{31$_6$}$$

3. DEVELOPMENT OF THE PERTURBATION

Let f_1 denote the true anomaly of the Moon; then its position vector $\underset{\sim}{x}_1$ has for components

$$x_1 = r_1[\cos(f_1+g_1)\cos h_1 - \sin(f_1+g_1)\sin h_1 \cos I_1], \qquad (32_1)$$

$$y_1 = r_1[\cos(f_1+g_1)\sin h_1 + \sin(f_1+g_1)\cos h_1 \cos I_1], \qquad (32_2)$$

$$z_1 = r_1\sin(f_1+g_1)\sin I_1, \qquad (32_3)$$

and the elongation function s defined in (11) turns out to be the function

$$s = \cos[(f_1-\ell_1) - (f_2-\ell_2) + D_1]\cos^2 \tfrac{1}{2} I_1 + \cos[(f_1-\ell_1) + (f_2-\ell_2) + 2F_1 - D_1]\sin^2 \tfrac{1}{2} I_1. \qquad (33)$$

The systematic development of s may proceed as follows.

(i) One obtains the d'Alembert series

$$\cos(f-\ell) = 1 + \sum_{j\geq 1}\left(\sum_{k\geq 0} e_{j,k}e^{2k}\right)e^j\cos j\ell, \qquad (34_1)$$

$$\sin(f-\ell) = \sum_{j\geq 1}\left(\sum_{k\geq 0} s_{j,k}e^{2k}\right)e^j\sin j\ell. \qquad (34_2)$$

Several algorithms have already been proposed to generate such series by computer (Deprit and Rom 1967, Broucke 1969); we propose to come back to this elementary but important problem in the next installment.

(ii) In the series (34) applied to the Moon, i.e. for $f = f_1$, $\ell = \ell_1$, and $e = e_1$, we make the substitutions

$$e_1^{2n} = (E_1^2 - \frac{1}{4} E_1^4)^n,$$

$$e_1^{2n+1} = (E_1^2 - \frac{1}{4} E_1^4)\varepsilon_1^n$$

where ε_1 is the expansion of

$$e_1 = E_1(1 - \frac{1}{4} E_1^2)^{1/2}$$

in powers of E_1. These substitutions produce a d'Alembert series of the type

$$\cos(f_1-\ell_1) = 1 + \sum_{j\geq 1}\left(\sum_{k\geq 0} c_{j,k}E_1^{2k}\right)E_1^j \cos j\ell_1,$$

$$\sin(f_1-\ell_1) = \sum_{j\geq 1}\left(\sum_{k\geq 0} s_{j,k}E_1^{2k}\right)E_1^j \sin j\ell_1.$$

(iii) Then, by straightforward multiplication, we calculate the products

$$M_1 = \cos(f_1-\ell_1)\cos(f_2-\ell_2),$$

$$M_2 = \cos(f_1-\ell_1)\sin(f_2-\ell_2),$$

$$M_3 = \sin(f_1-\ell_1)\cos(f_2-\ell_2),$$

$$M_4 = \sin(f_1-\ell_1)\sin(f_2-\ell_2),$$

which come out as d'Alembert series in the pairs (E_1,ℓ_1) and (e_2,ℓ_2).

(iv) As the last intermediate step, we produce

$$\cos[(f_1-\ell_1) - (f_2-\ell_2) + D_1] = (M_1+M_4)\cos D_1 + (M_2-M_3)\sin D_1,$$

$$\cos[(f_1-\ell_1) + (f_2-\ell_2) + 2F_1 - D_1] = (M_1-M_4)\cos(2F_1-D_1) - (M_2+M_3)\sin(2F_1-D_1).$$

The first trigonometric function will come out of the straightforward multiplication as a series

$$\sum_{i_1\geq 0} \sum_{i_2\geq 0} \sum_{j_1} \sum_{j_2} c_{i_1,i_2}^{j_1,j_2} E_1^{i_1} e_2^{i_2} \cos(j_1\ell_1 + j_2\ell_2 \pm D_1)$$

with the d'Alembert characteristics

$$j_1 \equiv i_1 \pmod{2}, \qquad |j_1| \leq i_1,$$

$$j_2 \equiv i_2 \pmod{2}, \qquad |j_2| \leq i_2;$$

similarly the second product will be a series

$$\sum_{i_1\geq 0} \sum_{i_2\geq 0} \sum_{j_1} \sum_{j_2} c_{i_1,i_2}^{j_1,j_2} E_1^{i_1} e_2^{i_2} \cos(j_1\ell_1 + j_2\ell_2 \pm 2F_1 \mp D_1)$$

with the same d'Alembert characteristics.

According to (34), the elongation function s can now be assembled as a series in E_1, e_2; by means of the development

$$\sin^2 \tfrac{1}{2} I_1 = \tfrac{1}{4} J_1^2 (1 - \tfrac{1}{2} E_1^2)^{-1} = \tfrac{1}{4} J_1^2 \sum_{j\geq 0} \frac{1}{2^j} E_1^{2j}, \tag{35}$$

it can be reworked as the series

$$s = \sum_{i_1\geq 0}\sum_{i_2\geq 0}\sum_{i_3=0,1}\sum_{j_1}\sum_{j_2}\sum_{j_3=0,1} c^{i_1,i_2,i_3}_{j_1,j_2,j_3} E_1^{i_1} e_2^{i_2} J_1^{2i_3} \cos(j_1\ell_1+j_2\ell_2+2j_3F_1\pm D_1) \tag{36}$$

with the d'Alembert characteristics

$$|j_1| \leq i_1, \quad j_1 \equiv i_1 \pmod{2},$$

$$|j_2| \leq i_2, \quad j_2 \equiv i_2 \pmod{2},$$

$$|j_3| \leq i_3.$$

At this point we depart from the suggestions made by Delaunay (1860, p. 17 to 20): a systematic procedure is in order if the development of the perturbation is to be carried by computer.

From Kepler's third law applied to Sun and Moon, we have that

$$\frac{k^2 m_2 a_1^2}{a_2^3} = \frac{k^2(m_0+m_1)}{a_1} m^2\sigma_2. \tag{37}$$

Delaunay (1860, p. 20) decides at this point to take $\sigma_2 = 1$. Actually since the algebraic manipulations are carried by computer, there is no need for erasing the symbol σ_2; its presence in the calculations is very useful for preliminary checks and for tracking the origin of the terms appearing in an operation. We also put

$$r_1 = a_1\rho_1, \tag{38$_1$}$$

$$r_2 = a_2\rho_2, \tag{38$_2$}$$

$$\phi_n = \frac{(1-\sigma_1)^{n-1}-(-\sigma_1)^{n-1}}{(1-2\sigma_1)^{n-2}} \quad \text{for} \quad n \geq 2. \tag{38$_3$}$$

With these conventions, the perturbation function $\mathcal{P}$ given by (17) can be slightly reworked to appear in the form

$$\mathcal{P} = -\frac{k^2(m_0+m_1)}{a_1}\sigma_2 m^2 \sum_{j\geq 2} \phi_j \alpha_1^{j-2} \frac{\rho_1^j}{\rho_2^{j+1}} P_j(s). \tag{39}$$

In this manner, $\mathcal{P}$ is more akin to de Pontécoulant's perturbation than to Delaunay's series: the parallactic parameter α_1 does not enter explicitly the part of $\mathcal{P}$ corresponding to $j = 2$ (Brown 1896, pp. 7 and 82).

Let us examine the mass functions ϕ_j; to this effect, we introduce the correction factor

$$c_1 = \frac{\sigma_1(1-\sigma_1)}{(1-2\sigma_1)^2} = \frac{m_0 m_1}{(m_0-m_1)^2} . \tag{40}$$

By putting also

$$\beta_1 = (1+4c_1)^{1/2} \tag{41}$$

and inverting the definition (40), we recognize easily that

$$\sigma_1 = \frac{1-\beta_1}{2\beta_1}, \qquad 1-\sigma_1 = \frac{1+\beta_1}{2\beta_1}, \qquad 1-2\sigma_1 = \frac{1}{\beta_1}$$

which implies that, for $j \geq 2$,

$$\phi_j = \frac{1}{2^{j-1}} \sum_{0\leq k\leq j-2} \binom{j-1}{k+1}[1 + (-1)^k]\beta_1^k. \tag{42}$$

Since they constitute polynomials in the even powers of β_1, the mass functions ϕ_j are polynomials in the correction factor c_1. The first eight ones are as follows:

$$\Phi_2 = 1,$$
$$\Phi_3 = 1,$$
$$\Phi_4 = 1 + c_1,$$
$$\Phi_5 = 1 + 2c_1,$$
$$\Phi_6 = 1 + 3c_1 + c_1^2,$$
$$\Phi_7 = 1 + 4c_1 + 3c_1^2,$$
$$\Phi_8 = 1 + 5c_1 + 6c_1^2 + c_1^3,$$
$$\Phi_9 = 1 + 6c_1 + 10c_1^2 + 4c_1^3.$$

The tradition in Analytical Lunar Theories has been to drop from $\mathcal{P}$ the terms having c_1 as a factor (Brown 1896, p. 8). For the time being we shall refrain from doing so.

Let us now establish a recursive procedure to expand the terms

$$\mathcal{P}_j = \frac{\rho_1^j}{\rho_2^{j+1}} P_j(s). \qquad (j \geq 0) \tag{43}$$

From a library of elliptic expansions in the problem of two bodies, we produce the three d'Alembert series

$$\mathcal{P}_0 = \frac{1}{\rho_2} = 1 + \sum_{j\geq 1}\left(\sum_{k\geq 0} c_{j,k} e_2^{2k}\right) e_2^j \cos j\ell_2, \tag{44$_1$}$$

$$\rho_1 = 1 + \frac{1}{2}(E_1^2 - \frac{1}{4} E_1^4) + \sum_{j\geq 1}\left(\sum_{k\geq 0} c_{j,k} E_1^{2k}\right) E_1^j \cos j\ell_1, \tag{44$_2$}$$

$$\frac{\rho_1}{\rho_2} = \sum_{i_1\geq 0}\ \sum_{i_2\geq 0}\ \sum_{j_1}\ \sum_{j_2} c_{j_1,j_2}^{i_1,i_2} E_1^{i_1} e_2^{i_2} \cos(j_1\ell_1 + j_2\ell_2), \tag{44$_3$}$$

hence, by straightforward multiplication of (36), (44_2) and (44_3) the d'Alembert series

$$\mathcal{P}_1 = s\frac{\rho_1}{\rho_2}\mathcal{P}_0 = \sum_{i_1\geq 0}\sum_{i_2\geq 0}\sum_{i_3=0,1}\sum_{j_1}\sum_{j_2}\sum_{j_3} c^{i_1,i_2,i_3}_{j_1,j_2,j_3} E_1^{i_1} e_2^{i_2} J_1^{2i_3} \tag{44_4}$$

$$\cos(j_1\ell_1 + j_2\ell_2 + 2j_3F_1 \pm D_1).$$

Remembering the recursive relation

$$(j+1)P_{j+1}(s) = (2j+1)sP_j(s) - jP_{j-1}, \qquad (j \geq 1)$$

between Legendre's polynomials, we infer the recurrence

$$(j+1)\mathcal{P}_{j+1}(s) = \frac{\rho_1}{\rho_2}\left[(2j+1)s\mathcal{P}_j - j\frac{\rho_1}{\rho_2}\mathcal{P}_{j-1}\right]. \qquad (j \geq 1)$$

It is now a matter of straightforward multiplications and linear combinations of Poisson series to obtain the functions $\mathcal{P}_n$ in the form

$$\mathcal{P}_n = \sum_{i_2\geq 0}\sum_{i_3\geq 0}\sum_{i_4\geq 0} E_1^{i_2} e_2^{i_3} J_1^{2i_4} \sum_{j_1}\sum_{j_2}\sum_{j_3}\sum_{j_4} c^{n,i_2,i_3,i_4}$$

$$\cos(j_1D_1 + j_2\ell_1 + j_3\ell_2 + j_4F_1)$$

with the d'Alembert characteristics

$$\begin{aligned}
j_1 &\equiv n \pmod{2}, & |j_1| &\leq n,\\
j_2 &\equiv i_2 \pmod{2}, & |j_2| &\leq i_2,\\
j_3 &\equiv i_3 \pmod{2}, & |j_3| &\leq i_3,\\
j_4 &\equiv i_4 \pmod{2}, & |j_4| &\leq i_4 \leq n.
\end{aligned}$$

Substitution of these terms into (39) delivers the perturbation

$$\mathcal{P} \equiv \mathcal{P}(\alpha_1, E_1, J_1, e_2;\ \frac{k^2(m_0+m_1)}{a_1}, m, \sigma_2, c_1 : D_1, F_1, \ell_1, \ell_2)$$

as a series in the powers of the modified parallax parameter α_1, the Moon's eccentricity parameter E_1, its inclination parameter J_1 and the Sun's eccentricity e_2. The coefficients in this series are finite trigonometric sums in the arguments D_1, F_1, ℓ_1 and ℓ_2; the pairs (α_1, D_1), (E_1, ℓ_1), (e_2, ℓ_2), (J_1, F_1) present the d'Alembert characteristics listed for any $\mathcal{P}_n$. The coefficients of the trigonometric sums are polynomials in the mass correction c_1, each of them to be multiplied by the common factor

$$\frac{k^2(m_0+m_1)}{a_1} m^2 \upsilon_2 .$$

4. COMPARISON WITH DELAUNAY

The development of the perturbation has been performed automatically on an IBM 360. The output is a deck of punched cards in hexadecimal code formated to serve as input in the further construction of the Analytical Ephemeris.

The main program and its subroutines have been tested by reconstructing Delaunay's expansion. Thus, as it is done in the *Théorie du Mouvement de la Lune*, we eliminated the mass ratios by assuming that

$$\sigma_2 = 1, \qquad \sigma_1 = 0, \qquad c_1 = 0;$$

also we used the eccentricity e_1 of the Moon instead of the function E_1, and the variable

$$\gamma_1 = \sin \frac{1}{2} I_1$$

instead of the function J_1.

Comparison term by term between Delaunay's text and the computer listings has been made by Mrs. Deprit-Bartholomé.

In the expansion we had retained all terms whose characteristic

$$\alpha^{j_1} e_1^{j_2} \gamma_1^{j_3} e_2^{j_4}$$

is such that

$$2j_1 + j_2 + j_3 + j_4 \leq 10.$$

Delaunay has explained why he did not collect all of these terms. It may be of interest to learn by how much Delaunay's considerations on the

"order" of e_2 have simplified his perturbation function. We enter in Table I the 103 missing terms having α^2 as a factor (they all have characteristics such that $j_2 + j_3 + j_4 = 6$) and in Table II the 19 missing terms having α^3 as a factor (their characteristics are such that $j_2 + j_4 = 4$ and $j_3 = 0$). The trigonometric arguments are listed in the manner Delaunay ordered them, except for the last 29 entries in Table I and the last 8 ones in Table II which are not to be found in Delaunay's perturbation. Some of the terms appear in Tables I and II with significantly large coefficients; one may wonder if Delaunay was justified in omitting them.

All other terms that are common to Delaunay's perturbation and ours agree in their coefficients.

Now Delaunay's expansion has been recovered by Barton (1967), Broucke (1969) and Jefferys (1970); each of these authors report agreement with their codes. Therefore we can presume that our programs are operational and our development is correct.

Table 1. Terms in α^2 omitted by Delaunay.

Characteristics				Arguments				Numerators	Denominators
α	e_1	γ_1	e_2	ℓ_1	ℓ_2	D_1	F_1		
2	0	0	6	0	0	0	0	35	64
2	0	0	6	0	0	2	0	-35	384
2	0	2	4	0	0	2	0	-39	32
2	2	0	4	0	0	2	0	-195	128
2	2	0	4	2	0	0	0	-15	64
2	1	0	5	1	1	0	0	-261	256
2	1	2	3	1	1	0	0	81	16
2	3	0	3	1	1	0	0	27	256
2	1	0	5	1	-1	0	0	-261	256
2	1	2	3	1	-1	0	0	81	16
2	3	0	3	1	-1	0	0	27	256
2	0	0	6	0	2	0	0	141	128
2	2	0	4	2	-2	2	0	39	64
2	1	0	5	1	-1	2	0	1467	512
2	1	2	3	1	-1	2	0	369	32
2	3	0	3	1	-1	2	0	7011	512
2	5	0	1	1	-1	2	0	2247	512
2	1	0	5	1	-1	2	0	-5	512
2	1	2	3	1	-1	2	0	-3	32
2	3	0	3	1	-1	2	0	-57	512
2	1	0	5	-1	-1	2	0	-4401	512
2	1	2	3	-1	-1	2	0	-1107	32
2	3	0	3	-1	-1	2	0	-4797	512
2	1	0	5	-1	1	2	0	15	512
2	1	2	3	-1	1	2	0	9	32
2	3	0	3	-1	1	2	0	39	512
2	0	0	6	0	-2	2	0	601	64
2	0	2	4	0	-2	2	0	115	4
2	2	0	4	0	-2	2	0	575	16
2	0	2	4	0	0	0	2	45	16
2	3	0	3	3	1	0	0	-27	256
2	3	0	3	3	-1	0	0	-27	256
2	2	0	4	2	2	0	0	-7	32
2	2	0	4	2	-2	0	0	-7	32
2	1	0	5	1	3	0	0	-393	512
2	1	2	3	1	3	0	0	159	16
2	3	0	3	1	3	0	0	53	256
2	1	0	5	1	-3	0	0	-393	512
2	1	2	3	1	-3	0	0	159	16
2	3	0	3	1	-3	0	0	53	256
2	0	0	6	0	4	0	0	129	130
2	3	0	3	3	-1	2	0	-3075	512
2	3	0	3	3	-1	2	0	25	512

Table I (continued)

Characteristics				Arguments				Numerators	Denominators
α	e_1	γ_1	e_2	ℓ_1	ℓ_2	D_1	F_1		
2	3	0	3	-3	-1	2	0	861	512
2	3	0	3	-3	1	2	0	-7	512
2	2	0	4	2	-2	2	0	-115	8
2	1	0	5	1	-3	2	0	-32525	1024
2	1	2	3	1	-3	2	0	-845	32
2	3	0	3	1	-3	2	0	-16055	512
2	1	0	5	1	-1	2	0	11	1024
2	1	2	3	1	-1	2	0	-1	32
2	3	0	3	1	-1	2	0	-19	512
2	1	0	5	-1	-3	2	0	97575	1024
2	1	2	3	-1	-3	2	0	2535	32
2	3	0	3	-1	-3	2	0	10985	512
2	1	0	5	-1	3	2	0	-33	1024
2	1	2	3	-1	3	2	0	3	32
2	3	0	3	-1	3	2	0	13	512
2	0	0	6	0	-4	2	0	-41481	640
2	0	2	4	0	-4	2	0	-1599	32
2	2	0	4	0	-4	2	0	-7995	128
2	0	0	6	0	4	2	0	7	320
2	0	2	4	0	4	2	0	-1	16
2	2	0	4	0	4	2	0	-5	64
2	1	2	3	1	1	0	2	81	32
2	1	2	3	1	-1	0	2	81	32
2	1	2	3	-1	1	0	2	-243	32
2	1	2	3	-1	-1	0	2	-243	32
2	0	2	4	0	2	0	2	21	8
2	0	2	4	0	-2	0	2	21	8
2	1	2	3	1	-1	2	-2	369	32
2	1	2	3	1	1	2	-2	-3	32
2	1	2	3	-1	-1	2	-2	369	32
2	1	2	3	-1	1	2	-2	-3	32
2	1	2	3	-1	3	0	2	159	32 (*)
2	1	2	3	-1	-3	0	2	-477	32
2	1	2	3	1	3	2	-2	-1	32
2	1	2	3	-1	-3	2	-2	-845	32
2	3	0	3	3	3	0	0	-53	256
2	3	0	3	3	3	2	0	25	1536
2	3	0	3	-3	-3	2	0	-5915	1536
2	3	0	3	3	-3	0	0	-53	256
2	3	0	3	-3	3	2	0	-7	1536
2	3	0	3	3	-3	2	0	21125	1536
2	1	2	3	-1	3	0	2	-477	32
2	1	2	3	1	-3	0	2	159	32
2	1	2	3	-1	3	2	-2	-1	32
2	1	2	3	1	-3	2	-2	-845	32

(*) From here on the arguments do not enter Delaunay's perturbation.

Table I (continued)

Characteristics				Arguments				Numerators	Denominators
α	e_1	γ_1	e_2	ℓ_1	ℓ_2	D_1	F_1		
2	0	2	4	0	4	0	2	231	32
2	0	2	4	0	-4	0	2	231	32
2	2	0	4	2	4	0	0	-77	128
2	2	0	4	2	4	2	0	1	32
2	2	0	4	2	-4	0	0	-77	128
2	2	0	4	2	-4	2	0	1599	64
2	1	0	5	1	5	0	0	-1733	512
2	1	0	5	1	5	2	0	243	5120
2	1	0	5	-1	-5	2	0	-685041	5120
2	1	0	5	1	-5	0	0	-1773	512
2	1	0	5	-1	5	2	0	-729	5120
2	1	0	5	1	-5	2	0	228347	5120
2	0	0	6	0	6	0	0	3167	640
2	0	0	6	0	6	2	0	1	15
2	0	0	6	0	-6	2	0	73369	960

Table II. Terms in α^3 omitted by Delaunay.

Characteristics				Arguments				Numerators	Denominators	
α	e_1	γ_1	e_2	l_1	l_2	D_1	F_1			
3	0	0	4	0	0	1	0	717	512	
3	0	0	4	0	0	3	0	2115	512	
3	1	0	3	1	-1	1	0	-33	64	
3	1	0	3	1	1	1	0	-15	32	
3	0	0	4	0	-2	1	0	117	128	
3	1	0	3	1	-1	3	0	-165	8	
3	1	0	3	1	1	3	0	95	64	
3	1	0	3	-1	-1	3	0	495	8	
3	1	0	3	-1	1	3	0	-225	64	
3	0	0	4	0	-2	3	0	15325	384	
3	0	0	4	0	2	3	0	5	384	
3	1	0	3	3	1	1	0	-23	64	(*)
3	1	0	3	-1	-3	3	0	-7335	64	
3	1	0	3	1	-3	3	0	2445	64	
3	1	0	3	1	-3	1	0	-77	32	
3	0	0	4	0	4	1	0	1029	1024	
3	0	0	4	0	4	3	0	5	3072	
3	0	0	4	0	-4	3	0	177065	3072	
3	0	0	4	0	-4	1	0	8865	1024	

(*) From here on the arguments do not enter Delaunay's perturbation.

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